Advanced automation and control

21-06-2019

Exercise 1

$$\max_{x_1,\delta_1,\delta_2} \quad 0.5x_1 + 2\delta_1 + 4\delta_2$$
$$-x_1 + \delta_2 \geq 3\delta_1$$
$$\delta_1, \delta_2 \in \{0,1\}$$

$\longrightarrow$

Standard form ($x_1 = x_1^+ - x_1^-$)

$$\max \quad 0.5x_1^+ - 0.5x_1^- + 2\delta_1 + 4\delta_2$$
$$3\delta_1 + x_1^+ - x_1^- - \delta_2 + s_1 = 0$$
$$\delta_1, \delta_2 \in \{0,1\}$$
$$x_1^+, x_1^-, s_1 \geq 0$$

Relaxation node 0

$$\max \quad 0.5x_1^+ - 0.5x_1^- + 2\delta_1 + 4\delta_2$$
$$3\delta_1 + x_1^+ - x_1^- - \delta_2 + s_1 = 0$$
$$\delta_1 + s_2 = 1$$
$$\delta_2 + s_3 = 1$$
$$x_1^+, x_1^-, \delta_1, \delta_2, s_1, s_2, s_3 \geq 0$$

Phase 1

$$\min \quad y_1 + y_2 + y_3$$
$$3\delta_1 + x_1^+ - x_1^- - \delta_2 + s_1 + y_1 = 0$$
$$\delta_1 + s_2 + y_2 = 1$$
$$\delta_2 + s_3 + y_3 = 1$$
$$x_1^+, x_1^-, \delta_1, \delta_2, s_1, s_2, s_3, y_1, y_2, y_3 \geq 0$$

Now we start with the tableau. We follow the rules sent via mail. First the one of the variables order: $x_1^-, x_1^+, \delta_1, \delta_2, s_1, s_2, s_3, y_1, y_2, y_3$ - Also if one of the equalities is $=0$ we keep it so to have "$+s_1$" and not "$-s_1$". Please report always the variables in the basis and the aux-

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3	y_1	y_2	y_3
	0	0	0	0	0	0	0	0	1	1	1
s_1	0	-1	1	3	-1	1	0	0	1	0	0
s_2	1	0	0	1	0	0	1	0	0	1	0
s_3	1	0	0	0	1	0	0	1	0	0	1

Since all the coeff. in the first row are ≥ 0 and we are minimizing Phase 1 is over. Note that s_1, s_2, s_3 (and no y_i) are in the basis and the optimal cost is 0. Therefore Phase 1 is ok and we can continue with Phase 2

Phase 2

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3	
	0	-0.5	0.5	2	4	0	0	0	-0.5AUX
s_1	0	-1	1	3	-1	1	0	0	= AUX
s_2	1	0	0	1	0	0	1	0	
s_3	1	0	0	0	1	0	0	1	

$\longrightarrow$

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3	
	0	0	0	0.5	4.5	-0.5	0	0	-0.5 AUX
x_1^+	0	-1	1	3	-1	1	0	0	
s_2	1	0	0	1	0	0	1	0	- AUX
s_3	1	0	0	0	1	0	0	1	

AUX [0 ; -1/3 1/3 1 -1/3 1/3 0 0]

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3	
	0	1/6	-1/6	0	14/3	-2/3	0	0	-1/6 AUX
δ_1	0	-1/3	1/3	1	-1/3	1/3	0	0	+1/3 AUX
δ_2	1	1/3	-1/3	0	1/3	-1/3	1	0	
s_3	1	0	0	0	1	0	0	1	

AUX [3 ; 1 -1 0 1 -1 3 0]

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3	
	-1/2	0	0	0	9/2	-1/2	-1/2	0	-9/2 AUX
δ_1	1	0	0	1	0	0	1	0	
x_1^-	3	1	-1	0	1	-1	3	0	- AUX
s_3	1	0	0	0	1	0	0	1	= AUX

		x_1^-	x_1^+	δ_1	δ_2	s_1	s_2	s_3
	-5	0	0	0	0	-1/2	-1/2	-9/2
δ_1	1	0	0	1	0	0	1	0
x_1^-	2	1	-1	0	0	-1	3	-1
δ_2	1	0	0	0	1	0	0	1

OK! Phase 2 is over

Optimal cost: +5

Optmal value optimization variables:

$$x = \begin{bmatrix} x_1^- \\ x_1^+ \\ \delta_1 \\ \delta_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since δ_1 and δ_2 are binaries, the optmal solution of relaxation of node 0 is the optmal solution of the MILP!

loreover, the value of he original x_1 is

$$x_1 = x_1^+ - x_1^- = 0 - 2 = -2$$

Exercise 2

$\delta_{T1} = 1$ if Doe takes train from ZH to MI

$\delta_{B1} = 1$ " " " blabla car " " " "

$\delta_{C} = 1$ " " " car (his own) - One binary for ZH-MI and MI-ZH since if he goes by car he has to come back by car

$\delta_{T2} = 1$ " " " train from MI-ZH

$\delta_{B2} = 1$ " " " blabla car " " "

$\delta_{F} = 1$ " " " plane from LI-MI

$= 0$ " " " " " LI-ZH

$\delta_{P} = 1$ " the overall travel time exceeds 12 hours

x continuous variable containing the overall travel time

$$x = 4\delta_{T1} + 3.5\delta_{B1} + 6\delta_C + 4\delta_{T2} + 3.5\delta_{B2} + 6.5(1-\delta_F) + 6\delta_F$$

Constraints

$\delta_{T1} + \delta_{B1} + \delta_C = 1$ He has to go to Milan (only one transport)

$\delta_{T2} + \delta_{B2} + \delta_C = \delta_F$ He needs this transport only if he goes back to MI

Cost function

$$\min\ 19\delta_{T1} + 25\delta_{B1} + 70\delta_C + 19\delta_{T2} + 25\delta_{B2} + 300(1-\delta_F) + 250\delta_F + 20(x-12)\cdot\delta_P$$

We need to relate δ_P to x and get rid of the bilinearity

$\delta_P = 1 \leftrightarrow x > 12$

$\delta_P = 1 \rightarrow x - 12 > 0 \qquad x - 12 \geq \varepsilon + (L-\varepsilon)(1-\delta_P)$ ①

$\delta_P = 0 \rightarrow x - 12 \leq 0 \qquad x - 12 \leq U\delta_P$ ②

$L \leq x - 12 \leq U$

$L = \underset{\text{bla-bla}}{3.5} + \underset{\text{fly to ZH}}{6.5} - 12 = -2$

$U = \underset{\text{train}}{4} + \underset{\text{train}}{4} + \underset{\text{fly to MI}}{6} - 12 = 2$

Reordering variables

- $-x - \delta_P(\underset{\uparrow}{L-\varepsilon}) \leq -12 - (L-\varepsilon) - \varepsilon = -12 - L = -10$ $(-2-\varepsilon)$
- $x - U\delta_P \leq 12 \rightarrow x - 2\delta_P \leq 12.$

Bilinearity $x\delta_P = z$ $\qquad L \le x \le U$ $\quad (L = 10,\ U = 14)$

$$\begin{cases} z \le U\delta_P \\ z \ge L\delta_P \\ z \le x - L(1-\delta_P) \\ z \ge x - U(1-\delta_P) \end{cases} \qquad \begin{aligned} & z - 14\delta_P \le 0 \\ & 10\delta_P - z \le 0 \\ & z - x - 10\delta_P \le -10 \\ & -z + x + 14\delta_P \le 14 \end{aligned}$$

$$\min\ 19\delta_{T1} + 25\delta_{B1} + 70\delta_C + 19\delta_{T2} + 25\delta_{B2} - 50\delta_F + 20z - 240\delta_P\ (+300)$$

subject to

$$\begin{aligned} & z - 14\delta_P \le 0 \\ & -z + 10\delta_P \le 0 \\ & z - x - 10\delta_P \le -10 \\ & -z + x + 14\delta_P \le 14 \\ & \delta_{T1} + \delta_{B1} + \delta_C = 1 \\ & \delta_{T2} + \delta_{B2} + \delta_C - \delta_F = 0 \\ & -x + (2+\varepsilon)\delta_P \le -10 \\ & x - 2\delta_P \le 12 \\ & x - 4\delta_{T1} - 3.5\delta_{B1} - 6\delta_C - 4\delta_{T2} - 3.5\delta_{B2} + 0.5\delta_F = 6.5 \end{aligned}$$

The problem can be rewritten as $\min\ c^T x$

subj to $A_I x \le b_I$ $\quad x \ge 0,\ \delta_{T1}, \delta_{B1}, \delta_C, \delta_{T2}, \delta_{B2}, \delta_F, \delta_P \in \{0,1\}$

$A_E x = b_E$

Let $x = [x\ z\ \delta_{T1}\ \delta_{B1}\ \delta_C\ \delta_{T2}\ \delta_{B2}\ \delta_F\ \delta_P]^T$, Then $c^T = [0\ 20\ 19\ 25\ 70\ 19\ 25\ -50\ -240]$

$$A_I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -14 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -10 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (2+\varepsilon) \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \qquad b_I = \begin{bmatrix} 0 \\ 0 \\ -10 \\ 14 \\ -10 \\ 12 \end{bmatrix}$$

Note that in order to solve the problem with the simplex one will have to convert it to standard form.

$$A_E = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 1 & 0 & -4 & -3.5 & -6 & -4 & -3.5 & 0.5 & 0 \end{bmatrix} \qquad b_E = \begin{bmatrix} 1 \\ 0 \\ 6.5 \end{bmatrix}$$

Exercise 3

$$\min\ 0.25x_1^2 + 9x_2^2 - 3x_1$$
$$x_1^2 + x_2^2 \le 10$$
$$x_1^2 + x_2^2 \ge 3$$

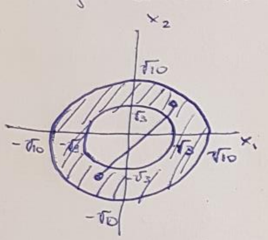
3.1 Let analyze the cost function. Since it contains quadratic terms I will rewrite it as

$$\min\ x^T Q x + c^T x \quad \rightarrow \quad Q = \begin{bmatrix} 0.25 & 0 \\ 0 & 9 \end{bmatrix} \quad c^T = [-3\ \ 0]$$

Q is positive definite (eigenvalues 9 and 0.25) and therefore the cost is CONVEX.

3.2. In order the optimization to be convex we need to <u>minimize</u> a <u>convex function</u> subject to a convex set of constraints. Therefore we need to check if the constraints are convex

3.3



The feasible set is not convex because if you pick any two points of it and draw the segment from one to the other this does not entirely lie inside the feasible set

For this reason the optimization is <u>not convex</u> (answer to 3.2)

Exercise 4

$$g_3(x_3) = J_3(x_3) = \begin{cases} 2 & x_3 = 1 \\ 1 & x_3 = 2 \\ 4 & x_3 = 3 \\ 4 & x_3 = 4 \end{cases}$$

$$J_2(x_2) = \begin{cases} \min(\overset{a}{1+4}, \overset{c}{3+4}) = 5 & x_2 = 1 \\ \overset{b}{5+2} = 7 & x_2 = 2 \\ \overset{a}{3+1} = 4 & x_2 = 3 \\ \min(\overset{b}{4+1}, \overset{c}{2+4}) = 5 & x_2 = 4 \end{cases}$$

$$\mu_2(x_2) = \begin{cases} a\ (1 \rightarrow 4) & x_2 = 1 \\ b\ (2 \rightarrow 1) & x_2 = 2 \\ a\ (3 \rightarrow 2) & x_2 = 3 \\ b\ (4 \rightarrow 2) & x_2 = 4 \end{cases}$$

$$J_1(x_1) = \begin{cases} \min(\underset{1\to 4}{\overset{a}{1} + 5}, \underset{1\to 3}{\overset{c}{3} + 4}) = 6 & x_1 = 1 \\ \overset{b}{5} + \underset{2\to 1}{5} = 10 & x_1 = 2 \\ \overset{a}{3} + \underset{3\to 2}{7} = 10 & x_1 = 3 \\ \min(\underset{4\to 2}{\overset{b}{4} + 7}, \underset{4\to 3}{\overset{c}{2} + 4}) = 6 & x_1 = 4 \end{cases}$$

$$\mu_1(x_1) = \begin{cases} a & x_1 = 1 \\ b & x_1 = 2 \\ a & x_1 = 3 \\ c & x_1 = 4 \end{cases}$$

$$J_0(x_0) = \begin{cases} \min(\underset{1\to 4}{\overset{a}{1} + 6}, \underset{1\to 3}{\overset{c}{3} + 10}) = 7 & x_0 = 1 \\ \overset{b}{5} + \underset{2\to 1}{6} = 11 & x_0 = 2 \\ \overset{a}{3} + \underset{3\to 2}{10} = 13 & x_0 = 3 \\ \min(\underset{4\to 2}{\overset{b}{4} + 10}, \underset{4\to 3}{\overset{c}{2} + 10}) = 12 & x_0 = 4 \end{cases}$$

$$\mu_0(x_0) = \begin{cases} a & x_0 = 1 \\ b & x_0 = 2 \\ a & x_0 = 3 \\ c & x_0 = 4 \end{cases}$$

$x_0 = 2 \longrightarrow \mu_0(x_0) = b \longrightarrow x_1 = 1 \rightarrow \mu_1(x_1) = a \rightarrow x_2 = 4 \rightarrow \mu_2(x_2) = b \rightarrow x_3 = 2$

Overall cost $J_0(2) = 11$